EQUALITY OF THE SPECTRAL AND DYNAMICAL DEFINITIONS OF REFLECTION

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ABSTRACT. For full-line Jacobi matrices, Schrödinger operators, and CMV matrices, we show that being reflectionless, in the sense of the well-known property of m-functions, is equivalent to a lack of reflection in the dynamics in the sense that any state that goes entirely to $x = -\infty$ as $t \to -\infty$ goes entirely to $x = \infty$ as $t \to \infty$. This allows us to settle a conjecture of Deift and Simon from 1983 regarding ergodic Jacobi matrices.

1. INTRODUCTION

In this paper, we discuss dynamics and spectral theory of whole-line Jacobi matrices, Schrödinger operators, and CMV matrices. In this introduction we focus on Jacobi matrices, that is, doubly infinite matrices,

$$J = \begin{pmatrix} \ddots & \ddots & \ddots & & & \\ & a_{-2} & b_{-1} & a_{-1} & & \\ & & a_{-1} & b_0 & a_0 & \\ & & & a_0 & b_1 & a_1 \\ & & & & \ddots & \ddots & \ddots \end{pmatrix} \qquad (1.1)$$

acting as operators on $\ell^2(\mathbb{Z})$. We suppose throughout that the Jacobi parameters, $\{a_n, b_n\}_{n=-\infty}^\infty$, are bounded.

We will sometimes need half-line Jacobi matrices given by

$$\begin{pmatrix} b_1 & a_1 & 0 & \dots \\ a_1 & b_2 & a_2 & \dots \\ 0 & a_2 & b_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad (1.2)$$

We call $\{a_n, b_n\}_{n=1}^{\infty}$ the Jacobi parameters for a half-line matrix and $\{a_n, b_n\}_{n=-\infty}^{\infty}$ the Jacobi parameters for a whole-line matrix.

We will call J *measure theoretically reflectionless* on a Borel set $\mathfrak{e} \subset \mathbb{R}$ if and only if for all n, the diagonal Green's function,

$$G_{nn}(\lambda + i0) = \lim_{\varepsilon \downarrow 0} \langle \delta_n, (J - \lambda - i\varepsilon)^{-1} \delta_n \rangle \qquad (1.3)$$

is pure imaginary, that is,

$$\mathrm{Re}\, G_{nn}(\lambda + i0) = 0 \qquad (1.4)$$

for Lebesgue a.e. $\lambda \in \mathfrak{e}$. Among the vast literature, we mention [6, 8, 9, 11, 12, 16, 20, 22, 25, 26, 27, 28, 29, 30, 31, 32, 36, 37, 41, 42, 43, 44, 45]. The name "reflectionless" is usually used without "measure theoretically" but we add this for reasons that will be clear shortly.

The notion first became commonly used in connection with solitons and has recently become especially important because of Remling's discovery [37] that right limits of half-line Jacobi matrices are measure theoretically reflectionless on Σ_{ac}, the essential support of the a.c. component of the half-line Jacobi matrix. The name comes from the fact that in the short-range case (i.e., $|a_n - 1| + |b_n| \to 0$ sufficiently rapidly as $|n| \to \infty$), the condition is equivalent to the time-independent reflection coefficient being zero on $\mathfrak{e}$.

There is a second notion of reflectionless operator depending on ideas of Davies–Simon [7]. For each $n \in \mathbb{Z}$, let χ_n^+ be the characteristic function of $[n, \infty)$ and χ_n^- of $(-\infty, n]$. We define

$$\mathcal{H}_\ell^+ = \left\{ \varphi \in \mathcal{H}_{\mathrm{ac}} \;\middle|\; \text{for all } n, \; \lim_{t \to -\infty} \|\chi_n^+ e^{-itJ} \varphi\| = 0 \right\} \qquad (1.5)$$

that is, states that, as $t \to -\infty$, are concentrated on the left. $\mathcal{H}_\ell^-$ is the same with $\lim_{t \to +\infty}$, and $\mathcal{H}_r^{\pm}$ are defined using χ_n^-. Here $\mathcal{H}_{\mathrm{ac}}$ is the a.c. subspace for J. We let P_{ac} be the projection onto $\mathcal{H}_{\mathrm{ac}}$, and let $P_{\ell,r}^{\pm}$ be the orthogonal projection onto $\mathcal{H}_{\ell,r}^{\pm}$, that is,

$$P_\ell^{\pm} = \operatorname*{s\text{-}lim}_{t \to \mp\infty} e^{itJ} \chi_0^- e^{-itJ} P_{\mathrm{ac}}(J) \qquad P_r^{\pm} = \operatorname*{s\text{-}lim}_{t \to \mp\infty} e^{itJ} \chi_0^- e^{-itJ} P_{\mathrm{ac}}(J)$$
$$(1.6)$$

Davies–Simon prove (they treat the analog for Schrödinger operators, but the argument is identical):

Theorem 1.1 ([7]). *We have ($\oplus$ = orthogonal direct sum)*

$$\mathcal{H}_{\mathrm{ac}} = \mathcal{H}_\ell^+ \oplus \mathcal{H}_r^+ \qquad (1.7)$$

$$= \mathcal{H}_\ell^- \oplus \mathcal{H}_r^- \qquad (1.8)$$

That is, any a.c. state is a sum of a state that moves entirely to the left as $t \to -\infty$ and one that moves to the right.

We call J *dynamically reflectionless* on a Borel set $\mathfrak{e}$ if and only if

$$P_{\mathfrak{e}} P_{\mathrm{ac}} = P_{\mathfrak{e}} \tag{1.9}$$

(here $P_{\mathfrak{e}}$ is the spectral projection for J) and

$$P_{\mathfrak{e}}[\mathcal{H}_{\ell}^{+}] = P_{\mathfrak{e}}[\mathcal{H}_{r}^{-}] \tag{1.10}$$

Before stating our main theorem, we want to define a third notion of reflectionless operator for reasons that will become clear momentarily. For any $n \in \mathbb{Z}$, let J_n^{+} be the Jacobi matrix obtained from dropping the row and column with b_n and keeping the lower right piece, that is, J_n^{+} is the one-sided Jacobi matrix with Jacobi parameters

$$b_{\ell}^{(n),+} = b_{n+\ell} \qquad a_{\ell}^{(n),+} = a_{n+\ell} \tag{1.11}$$

J_n^{-} has parameters

$$b_{\ell}^{(n),-} = b_{n+1-\ell} \qquad a_{\ell}^{(n),-} = a_{n-\ell} \tag{1.12}$$

Thus, if a_n is replaced by 0, the whole-line Jacobi matrix J breaks into a direct sum of J_n^{+} and a matrix unitarily equivalent to J_n^{-} after reordering the indices in inverse order.

For any half-line Jacobi matrix, J, we define its m-function by

$$m(z, J) = \langle \delta_1, (J - z)^{-1} \delta_1 \rangle \tag{1.13}$$

and for a whole-line Jacobi matrix,

$$m_n^{\pm}(z, J) = m(z, J_n^{\pm}) \tag{1.14}$$

These are related to the Green's function (1.3) by

$$G_{nn}(z) = -\frac{1}{a_n^2 m_n^{+}(z) - m_n^{-}(z)^{-1}} \tag{1.15}$$

We call a whole-line Jacobi matrix *spectrally reflectionless* on a Borel set $\mathfrak{e}$ if for a.e. $\lambda \in \mathfrak{e}$ and all n,

$$a_n^2 m_n^{+}(\lambda + i0) \overline{m_n^{-}(\lambda + i0)} = 1 \tag{1.16}$$

By (1.15), (1.16) implies $\operatorname{Re} G_{nn} = 0$, so

$$(1.16) \text{ for } \lambda \text{ and } n \Rightarrow (1.4) \text{ for } \lambda \text{ and } n$$

and so

$$\begin{aligned} &J \text{ is spectrally reflectionless on } \mathfrak{e} \Rightarrow \\ &J \text{ is measure theoretically reflectionless on } \mathfrak{e} \end{aligned} \tag{1.17}$$

Moreover, as we will see below,

$$(1.16) \text{ for } \lambda \text{ and one } n \Rightarrow (1.16) \text{ for } \lambda \text{ and all } n \tag{1.18}$$

This set of ideas is rounded out by the following theorem:

Theorem 1.2 (Gesztesy–Krishna–Teschl [11]; Sodin–Yuditskii [45])**.** *If* (1.4) *holds for a.e.* $\lambda \in \mathfrak{e}$ *and three consecutive values of* n, *then* (1.16) *holds for a.e.* $\lambda \in \mathfrak{e}$ *and all* n.

In particular, in (1.17), $\Rightarrow$ can be replaced by $\Leftrightarrow$. However, this is not true for CMV matrices [4].

Here is our main result:

Theorem 1.3. *For any whole-line Jacobi matrix* J *and Borel set* $\mathfrak{e}$ *of positive Lebesgue measure,* J *is spectrally reflectionless on* $\mathfrak{e}$ *if and only if it is dynamically reflectionless on* $\mathfrak{e}$.

This verifies a 25-year old conjecture of Deift–Simon [9], namely

Corollary 1.4. *The a.c. spectrum for two-sided ergodic Jacobi matrices is dynamically reflectionless.*

Proof. By Kotani theory [25, 38], such operators are spectrally reflectionless on the a.c. spectrum. $\square$

This is a special case of a more general result that we will prove concerning reflection probability. Let $\Sigma_{\mathrm{ac}}^{(2)}$ be the set of $\lambda \in \mathbb{R}$ where J has multiplicity 2, so automatically a.c. spectrum (see [21, 23, 24, 39]). $P_{\ell,r}^{\pm}$ commute with J, so they take $\mathrm{Ran}(P_{\Sigma_{\mathrm{ac}}^{(2)}}(J))$ to itself. J restricted to $\mathrm{Ran}(P_{\ell,r}^{\pm} P_{\Sigma_{\mathrm{ac}}^{(2)}}(J))$ is of multiplicity 1. Thus,

$$R = P_\ell^+ P_\ell^- P_\ell^+ \restriction \mathrm{Ran}(P_\ell^+ P_{\Sigma_{\mathrm{ac}}^{(2)}}(J)) \tag{1.19}$$

is a scalar function of J, and so there is a function $R(E)$ on $\Sigma_{\mathrm{ac}}^{(2)}$ so that

$$R = R(J) \restriction \mathrm{Ran}(P_\ell^+ P_{\Sigma_{\mathrm{ac}}^{(2)}}(J)) \tag{1.20}$$

As defined by Davies–Simon [7], $R(\lambda)$ is the *dynamic reflection probability*, the probability that a state of energy λ that comes in from the left at very negative times goes out on the left. There is a time-reversal symmetry, namely that one gets the same function, R, with $P_\ell^- P_\ell^+ P_\ell^- \restriction \mathrm{Ran}(P_\ell^-)$. Similarly, there is a left-right symmetry, so one gets the same function with $P_r^+ P_r^- P_r^+ \restriction \mathrm{Ran}(P_r^+)$.

Define the *spectral reflection probability* by (see Theorem 2.4 below for why this is a good definition)

$$\left| \frac{a_0^2 m_0^+(\lambda + i0)\, \overline{m_0^-(\lambda - i0)} - 1}{a_0^2 m_0^+(\lambda + i0) m_0^-(\lambda - i0) - 1} \right|^2 \tag{1.21}$$

We will prove

Theorem 1.5. $R(\lambda)$ *is given by* (1.21) *on* $\Sigma_{\mathrm{ac}}^{(2)}(J)$.

Theorem 1.5 implies Theorem 1.3 since

$$R(J) \restriction \mathfrak{e} = 0 \Leftrightarrow P_\ell^+ P_\mathfrak{e} = P_r^- P_\mathfrak{e} \tag{1.22}$$

and

$$(1.21) = 0 \Leftrightarrow (1.16) \text{ holds} \tag{1.23}$$

The various formulae involving $m_n^\pm$ are complicated, in part because the simple formulae are given by Weyl solutions. It pays to rewrite them here since the rewriting is critical to our proof.

We are interested in solutions of

$$a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1} = z u_n \tag{1.24}$$

For any $z \in \mathbb{C}_+ = \{z \mid \operatorname{Im} z > 0\}$, there are solutions $u_n^\pm(z)$ which are ℓ^2 at $\pm\infty$, unique up to a constant. We will normalize by

$$u_0^\pm = 1 \tag{1.25}$$

By general principles (see, e.g., [46, Chap. 2], though our notation is slightly different from his), for Lebesgue a.e. λ, $u_n^\pm(\lambda + i\varepsilon)$ has a limit as $\varepsilon \downarrow 0$, which we denote by $u_n^\pm(\lambda + i0)$ which solves (1.24) at λ.

$m^\pm$ can be expressed in terms of $u^\pm$ by ([46])

$$m_n^+(\lambda + i0) = -\frac{u_{n+1}^+(\lambda + i0)}{a_n u_n^+(\lambda + i0)} \tag{1.26}$$

$$m_n^-(\lambda + i0) = -\frac{u_n^-(\lambda + i0)}{a_n u_{n+1}^-(\lambda + i0)} \tag{1.27}$$

The Green's function, (1.3), which is symmetric, is given for $n \le m$ by

$$G_{nm}(\lambda + i0) = \frac{u_n^-(\lambda + i0) u_m^+(\lambda + i0)}{W(\lambda + i0)} \tag{1.28}$$

where

$$W(z) = a_n[u_{n+1}^+(z) u_n^-(z) - u_{n+1}^-(z) u_n^+(z)] \tag{1.29}$$

is n-independent.

From these formulae, (1.15) is immediate. Moreover, with the normalization $u_{n=0}^\pm = 1$, we see that (1.16) is equivalent to $u_{n=1}^+(\lambda + i0) = \overline{u_{n=1}^-(\lambda + i0)}$ which, by uniqueness of solutions, implies

$$u_n^+(\lambda + i0) = \overline{u_n^-(\lambda + i0)} \tag{1.30}$$

for all n. This explains why (1.18) holds. It shows that

$$J \text{ is spectrally reflectionless for } \lambda \in \mathfrak{e} \Leftrightarrow (1.30) \text{ for } \lambda \in \mathfrak{e} \tag{1.31}$$

The key to our proof of Theorem 1.3 (and also Theorem 1.5) will be

Almost-Theorem 1.6. $\operatorname{Ran}(P_\ell^+ P_{\Sigma_{\mathrm{ac}}^{(2)}})$ *is spanned by* $\{u_n^+(\lambda + i0) \mid \lambda \in \Sigma_{\mathrm{ac}}^{(2)}\}$ *and* $\operatorname{Ran}(P_r^+ P_{\Sigma_{\mathrm{ac}}^{(2)}})$ *by* $\{u_n^-(\lambda + i0) \mid \lambda \in \Sigma_{\mathrm{ac}}^{(2)}\}$.

We call this an almost-theorem because we are, for now, vague about what we mean by "span." The $u_n^\pm$ are only continuum eigenfunctions, so by span we will mean suitable integrals.

We can now understand why the almost-theorem will imply Theorem 1.3. By time-reversal invariance,

$$P_r^- = \overline{P_r^+} \tag{1.32}$$

Thus,

J is dynamically reflectionless for $\lambda \in \mathfrak{e} \Leftrightarrow \overline{P_r^+} P_{\Sigma_{\mathrm{ac}}^{(2)}} = P_\ell^+ P_{\Sigma_{\mathrm{ac}}^{(2)}}$
$$\tag{1.33}$$

and the almost-theorem says the right side is the same as (1.30).

For short-range perturbations of the free Jacobi matrix ($b_n \equiv 0$, $a_n \equiv 1$), the almost-theorem follows from suitable stationary phase/integration by parts ideas as noted in Davies–Simon [7]. Such methods cannot work for general Jacobi matrices where $\Sigma_{\mathrm{ac}}^{(2)}$ might be a positive measure Cantor set. What we will see is by replacing the limit

$$P_\ell^+ = \operatorname*{s-lim}_{t \to -\infty} e^{itJ} \chi_0^- e^{-itJ} P_{\mathrm{ac}}(J) \tag{1.34}$$

that Davies–Simon [7] use by an abelian limit, a simple calculation will yield the almost-theorem.

Section 2 proves all the above results for Jacobi matrices. Section 3 discusses (continuum) Schrödinger operators and Section 4 CMV matrices.

2. The Jacobi Case

In this section, we prove Almost-Theorem 1.6 and use it to prove Theorem 1.5, and thereby Theorem 1.3. To make sense of Almost-Theorem 1.6, we need to begin with an eigenfunction expansion. While this expansion can be viewed as a rephrasing of Section 2.5 of Teschl [46], it is as easy to establish it from first principles as to manipulate the results of [46] to the form we need. Our use of Stone's formula is similar to that of Gesztesy–Zinchenko [18].

Fundamental to this is the matrix for $\lambda \in \mathbb{R}$,

$$S(\lambda)_{nm} = \lim_{\varepsilon \downarrow 0} (2\pi i)^{-1} [(J - \lambda - i\varepsilon)^{-1} - (J - \lambda + i\varepsilon)^{-1}]_{nm} \tag{2.1}$$

defined for a.e. $\lambda \in \mathbb{R}$ and all n, m. We use S for "Stone" or "spectral" since Stone's formula (Thm. VII.13 of [34]) and the spectral theorem imply that for any φ, ψ of finite support on $\mathbb{Z}$ and any Borel set, $\mathfrak{e}$,

$$\langle \varphi, P_{\mathfrak{e}} P_{\mathrm{ac}} \psi \rangle = \int_{\lambda \in \mathfrak{e}} \left(\sum_{n,m} \bar{\varphi}_n \psi_m S(\lambda)_{nm} \right) d\lambda \tag{2.2}$$

Define for $\lambda \in \Sigma_{\text{ac}}^{(2)}$,

$$f_{\pm}(\lambda) = \pm \frac{a_0 \operatorname{Im}(u_1^{\mp}(\lambda + i0))}{\pi |W(\lambda + i0)|^2} \tag{2.3}$$

where $u_n^{\pm}$ is normalized by (1.25) and W is given by (1.29). This looks asymmetric in $\pm$, but

$$f_+(\lambda) = -\frac{a_{-1} \operatorname{Im}(u_{-1}^-(\lambda + i0))}{\pi |W(\lambda + i0)|^2} \tag{2.4}$$

$$= \frac{a_{-1}^2 \operatorname{Im}(m_{-1}^-(\lambda + i0))}{\pi |W(\lambda + i0)|^2} \tag{2.5}$$

while

$$f_-(\lambda) = \frac{a_0^2 \operatorname{Im}(m_0^+(\lambda + i0))}{\pi |W(\lambda + i0)|^2} \tag{2.6}$$

symmetric under reflection about $n = 0$. This makes it clear that

$$f_{\pm}(\lambda) > 0 \qquad \text{a.e. } \lambda \in \Sigma_{\text{ac}}^{(2)} \tag{2.7}$$

The key to our eigenfunction expansion is

$$S_{nm}(\lambda) = \overline{u_n^+(\lambda + i0)} \, u_m^+(\lambda + i0) f_+(\lambda) + \overline{u_n^-(\lambda + i0)} \, u_m^-(\lambda + i0) f_-(\lambda) \tag{2.8}$$

for all n, m and a.e. $\lambda \in \Sigma_{\text{ac}}^{(2)}$.

Theorem 2.1. (2.8) *holds for all n, m and a.e. $\lambda \in \Sigma_{\text{ac}}^{(2)}$.*

Proof. By general principles on limits of Stieltjes transforms, for a.e. $\lambda \in \Sigma_{\text{ac}}^{(2)}$, $\lim_{\varepsilon \downarrow 0} u_n^{\pm}(\lambda + i\varepsilon) = u_n^{\pm}(\lambda + i0)$ exists. We will prove (2.8) for such λ. It is easy to see that $S_{nm}(\lambda) = S_{mn}(\lambda)$, so it suffices to consider the case $n \leq m$.

By the resolvent formula, for $\operatorname{Im} z > 0$,

$$\pi S_{nm}(z) \equiv (2i)^{-1}[(J - z)^{-1} - (J - \bar{z})^{-1}]_{nm}$$

$$= (\operatorname{Im} z) \sum_k (J - \bar{z})_{nk}^{-1} (J - z)_{km}^{-1} \tag{2.9}$$

$$= (\operatorname{Im} z)|W(z)|^{-2}(t_{nm}^{(1)} + t_{nm}^{(2)} + t_{nm}^{(3)}) \tag{2.10}$$

by (1.28), where

$$t_{nm}^{(1)} = \left[\sum_{k \leq n} |u_k^-(z)|^2 \right] \overline{u_n^+(z)} \, u_m^+(z) \tag{2.11}$$

$$t_{nm}^{(2)} = \left[\sum_{k \geq m+1} |u_k^+(z)|^2 \right] \overline{u_n^-(z)} \, u_m^-(z) \tag{2.12}$$

$$t_{nm}^{(3)} = \left[\sum_{k=n+1}^{m} \overline{u_k^+(z)}\, u_k^-(z) \right] \overline{u_n^-(z)}\, u_m^+(z) \tag{2.13}$$

Because of the $\operatorname{Im} z$ in front of (2.10), $\lim(\operatorname{Im} z)t_{nm}^{(3)}(\lambda + iy) = 0$ since the limit exists (the sum is finite). Similarly, we can change the summation limits of the k sums in $t^{(1)}, t^{(2)}$ to any other finite value, since in the limit, finite sums multiplied by $\operatorname{Im} z$ go to zero. The result is

$$S_{nm}(\lambda+i0) = q^{(1)}(\lambda) \overline{u_n^+(\lambda + i0)}\, u_m^+(\lambda+i0) + q^{(2)}(\lambda) \overline{u_n^-(\lambda + i0)}\, u_m^-(\lambda+i0) \tag{2.14}$$

where

$$\pi q^{(1)}(\lambda) = \lim_{\varepsilon \downarrow 0} |W(\lambda + i0)|^{-2}\varepsilon \sum_{k \le -1} |u_k^-(\lambda + i\varepsilon)|^2 \tag{2.15}$$

$$\pi q^{(2)}(\lambda) = \lim_{\varepsilon \downarrow 0} |W(\lambda + i0)|^{-2}\varepsilon \sum_{k \ge 1} |u_k^+(\lambda + i0)|^2 \tag{2.16}$$

By the resolvent formula for J_0^+ and the analog of (1.28) (with the normalization (1.25)),

$$\operatorname{Im} m_0^+(z) = \operatorname{Im}(J_0^+ - z)_{11}^{-1}$$

$$= (\operatorname{Im} z) \sum_{k=1}^{\infty} (J_0^+ - \bar{z})_{1k}^{-1} (J_0^+ - z)_{k1}^{-1}$$

$$= (\operatorname{Im} z) a_0^2 \sum_{k=1}^{\infty} |u_k^+(z)|^2 \tag{2.17}$$

so

$$q^{(2)}(\lambda) = f_-(\lambda) \tag{2.18}$$

and similarly,

$$q^{(1)}(\lambda) = f_+(\lambda) \tag{2.19}$$

This proves (2.8). □

From (2.8), we immediately get an eigenfunction expansion.

Theorem 2.2. *For any $\varphi \in \ell^2(\mathbb{Z})$ of finite support, define*

$$\widehat{\varphi}_{\pm}(\lambda) = \sum_n \overline{u_n^{\pm}(\lambda)}\, \varphi_n \tag{2.20}$$

as functions on $\Sigma_{\mathrm{ac}}^{(2)}$. Then

$$\int_{\Sigma_{\mathrm{ac}}^{(2)}} [|\widehat{\varphi}_+(\lambda)|^2 f_+(\lambda) + |\widehat{\varphi}_-(\lambda)|^2 f_-(\lambda)]\, d\lambda = \|P_{\mathrm{ac}} P_{\Sigma_{\mathrm{ac}}^{(2)}} \varphi\|^2 \tag{2.21}$$

So $\widehat{}_\pm$ extend to continuous maps of $\ell^2(\mathbb{Z})$ to $L^2(\Sigma_{ac}^{(2)}, f_\pm \, d\lambda)$. Moreover, if $\widehat{} = (\widehat{}_+, \widehat{}_-)$, then

$$\widehat{(J\varphi)}_\pm(\lambda) = \lambda \widehat{\varphi}_\pm(\lambda) \tag{2.22}$$

For each n,

$$\int_{\Sigma_{ac}^{(2)}} |u_n^\pm(\lambda)|^2 f_\pm(\lambda) \, d\lambda \le 1 \tag{2.23}$$

In particular, for any

$$g = (g_+, g_-) \in L^2(\Sigma_{ac}^{(2)}, f_+ \, d\lambda) \oplus L^2(\Sigma_{ac}^{(2)}, f_- \, d\lambda) \equiv \mathcal{H}_J$$

and any n, we can define

$$\check{g}_n = \int g_+(\lambda) u_n^+(\lambda + i0) f_+(\lambda) \, d\lambda + \int g_-(\lambda) u_n^-(\lambda + i0) f_-(\lambda) \, d\lambda \tag{2.24}$$

$\check{g}$ lies in $\ell^2(\mathbb{Z})$, and for any $\varphi \in \ell^2$,

$$\langle \check{g}, \varphi \rangle = \langle g, \widehat{\varphi} \rangle \tag{2.25}$$

and

$$\widehat{\check{g}} = g \tag{2.26}$$

We have $\check{g} \in \mathrm{Ran}(P_{ac} P_{\Sigma_{ac}^{(2)}})$ and $\check{}$ is a bijection of this range and $\mathcal{H}_J$.

Proof. (2.21) is immediate from (2.2) and (2.8). (2.22) follows from a summation by parts and

$$\sum_m J_{nm} u_m^\pm(\lambda + i0) = \lambda u_n^\pm(\lambda + i0) \tag{2.27}$$

(2.23) comes from putting δ_n into (2.21).

By (2.23), the integrals in (2.24) converge for all $g \in \mathcal{H}_J$. For φ of finite support, (2.25) is an interchange of integration and finite sum. In particular, if χ_N is the characteristic function of $\{j \in \mathbb{Z} \mid |j| \le N\}$ and $\varphi = \chi_N \check{g}$, (2.25) implies

$$\sum_{|j| \le N} |\check{g}_j|^2 \le \|g\| \, \|\widehat{\varphi}\|$$

$$\le \|g\| \, \|\varphi\|$$

$$= \|g\| \left(\sum_{|j| \le N} |g_j|^2 \right)^{1/2} \tag{2.28}$$

so for all N,

$$\|\chi_N \check{g}\| \le \|g\| \tag{2.29}$$

so $\check{g} \in \ell^2$ and

$$\|\check{g}\| \le \|g\| \tag{2.30}$$

Thus, (2.25) extends to all φ by continuity.

By (2.21) and (2.22), $\widehat{}$ is a unitary spectral representation for $\tilde{J} = J \restriction \mathrm{Ran}(P_{\mathrm{ac}} P_{\Sigma_{\mathrm{ac}}^{(2)}})$ on $\mathrm{Ran}(\widehat{})$. Since $\tilde{J}$ has uniform multiplicity 2, $\mathrm{Ran}(\widehat{})$ must be all $\mathcal{H}_J$. It follows that $(\widehat{})(\widehat{})^* = \mathbf{1}$ on $\mathcal{H}_J$. Since $\widecheck{} = (\widehat{})^*$, this is (2.26). $\qquad\square$

We can now prove a precise version of Almost-Theorem 1.6. Let

$$\mathcal{H}_J^{\pm} = \{g \in \mathcal{H}_J \mid g_{\mp} = 0\} \tag{2.31}$$

and let $P^{\pm}$ be the projection in $\ell^2(\mathbb{Z})$ onto the image of $\mathcal{H}_J^{\pm}$ under $\widecheck{}$. Then

Theorem 2.3. *We have*

$$P_r^+ P_{\Sigma_{\mathrm{ac}}^{(2)}}(J) = P^- \tag{2.32}$$

$$P_\ell^+ P_{\Sigma_{\mathrm{ac}}^{(2)}}(J) = P^+ \tag{2.33}$$

$$P_r^- P_{\Sigma_{\mathrm{ac}}^{(2)}}(J) = \overline{P^-} \tag{2.34}$$

$$P_\ell^- P_{\Sigma_{\mathrm{ac}}^{(2)}}(J) = \overline{P^+} \tag{2.35}$$

Remark. Let C be complex conjugation on ℓ^2. By $\bar{A}$ we mean CAC.

Proof. We claim that it suffices to prove for $\varphi \in \mathrm{Ran}(P^+)$ that

$$P_\ell^+ \varphi = \varphi \tag{2.36}$$

for then, by reflection in $n = 0$, we see that for $\psi \in \mathrm{Ran}(P^-)$,

$$P_r^+ \psi = \psi \tag{2.37}$$

and

$$(P_\ell^+ + P_r^+) P_{\Sigma_{\mathrm{ac}}^{(2)}}(J) = P_{\Sigma_{\mathrm{ac}}^{(2)}}(J) = P^+ + P^- \tag{2.38}$$

implies (2.32)/(2.33). Since $\overline{e^{-itJ}} = e^{itJ}$, (2.34)/(2.35) then follow.

Clearly, it suffices to prove (2.36) for a dense set of $\varphi \in \mathrm{Ran}(P^+)$; equivalently, for a dense set of $g \in L^2(\Sigma_{\mathrm{ac}}^{(2)}, f_+ \, d\lambda)$, where

$$\varphi_n = \int g(\lambda) u_n^+(\lambda + i0) f_+(\lambda) \, d\lambda \tag{2.39}$$

By Egoroff's theorem, for a dense set of g, we can suppose $g \in L^\infty$, and for each fixed m, n, $G_{nm}(\lambda + ik^{-1}) \to G_{nm}(\lambda + i0)$ as $k \to \infty$, uniformly for $\lambda \in \mathrm{supp}(g)$. We henceforth assume these properties for g.

By (1.6) and an abelian theorem [35, Sect. XI.6, Lemma 5],

$$P_\ell^+ \varphi = \chi_0^- \varphi - i \lim_{t \to -\infty} \int_t^0 e^{isJ} [J, \chi_0^-] e^{-isJ} \varphi \, ds$$

$$= \chi_0^- \varphi - i \lim_{\varepsilon \downarrow 0} \int_{-\infty}^0 e^{\varepsilon s} e^{isJ} [J, \chi_0^-] e^{-isJ} \varphi \, ds$$

Since the limit exists, we can replace ε by $1/k$ and do the s integral

$$(P_\ell^+ \varphi)_n = \chi_0^-(n) \varphi_n - \lim_{k \to \infty} \sum_{m,n=-\infty}^\infty \int G_{nm} \left(\lambda + \frac{i}{k} \right)^{-1}$$
$$[J, \chi_0^+]_{m\ell} g(\lambda) u_\ell^+ (\lambda + i0) f_+(\lambda) \, d\lambda$$

But $[J, \chi_0^-]$ is rank two. In fact, $[J, \chi_0^+]_{m\ell} \neq 0$ only for $(m, \ell) = (0, 1)$ or $(1, 0)$, so the sum is finite, and by the uniform convergence of $G_{nm}(\lambda + \frac{i}{k})$ for $\lambda \in \mathrm{supp}(g)$ and $u_\ell^+ \in L^2(\mathbb{R}, f_+ \, d\lambda)$, we see that we can take the limit inside the integral. The result is

$$(P_\ell^+ \varphi)_n = \chi_0^-(n) \varphi_n - \int a_0 g(\lambda)$$
$$[G_{n1}(\lambda + i0) u_0^+(\lambda + i0) - G_{n0}(\lambda + i0) u_1^+(\lambda + i0)] f_+(\lambda) \, d\lambda$$
$$(2.40)$$

If $n > 0$, using (1.28),

$$a_0 [G_{n1}(\lambda + i0) u_0^+(\lambda + i0) - G_{n0}(\lambda + i0) u_1^+(\lambda + i0)]$$
$$= \frac{a_0 [u_0^+(\lambda + i0) u_1^-(\lambda + i0) - u_1^+(\lambda + i0) u_0^-(\lambda + i0)]}{W(\lambda)} u_n^+(\lambda + i0)$$
$$(2.41)$$

$$= u_n^+(\lambda + i0)$$

so (2.40) says

$$(P_\ell^+ \varphi)_n = \varphi_n \tag{2.42}$$

If $n \leq 0$, the $u_{0,1}^+$ in (2.41) becomes $u_{0,1}^-$ and u_n^+ becomes u_n^-, so the factor in [ ] is zero, and again (2.42) holds. $\square$

Remark. χ_0^- can be replaced by any χ_ℓ^-. So in the analog of (2.40) (where G_{n1}, G_{n0} become $G_{n\ell+1} G_{n\ell}$), one can even take ℓ to be n-dependent. Using this, one can use either the argument we used for $n > 0$ (by picking $\ell < n$) or for $n \leq 0$ (by picking $\ell \geq n$) rather than needing both calculations!

The above implies $P_\ell^+ P_{\mathfrak{e}} = P_r^- P_{\mathfrak{e}}$ if and only if for a.e. $\lambda \in \mathfrak{e}$, $u_n^+ = \overline{u_n^-}$, which holds if and only if, by (1.26)/(1.27), (1.16) holds for a.e. $\lambda \in \mathfrak{e}$. Thus, one has Theorem 1.3.

The following proves Theorem 1.5, and thereby completes the proofs of the results stated in Section 1.

Theorem 2.4. *For a.e.* $\lambda \in \Sigma_{\mathrm{ac}}^{(2)}$, *we can write*

$$u_n^+(\lambda + i0) = \alpha(\lambda) \overline{u_n^+(\lambda + i0)} + \beta(\lambda) \overline{u_n^-(\lambda + i0)} \qquad (2.43)$$

and the function R of (1.20) *is given by*

$$R(\lambda) = |\alpha(\lambda)|^2 \qquad (2.44)$$

Moreover, $R(\lambda)$ is given by (1.21).

Proof. For a.e. $\lambda \in \Sigma_{\mathrm{ac}}^{(2)}$, $\mathrm{Im}\, u_n^+(\lambda) < 0$, $\mathrm{Im}\, u_n^-(\lambda) > 0$, so $u^\pm(\lambda)$ are linearly independent solutions of $Ju = \lambda u$. It follows that (2.43) holds. If

$$\varphi = \int g(\lambda) u_n^+(\lambda + i0) f_+(\lambda)\, d\lambda \in \mathrm{Ran}(P_\ell^+) \qquad (2.45)$$

then (2.43) implies that

$$(P_\ell^- \varphi)_n = \int g(\lambda)\alpha(\lambda) \overline{u_n^+(\lambda + i0)}\, f_+(\lambda)\, d\lambda \qquad (2.46)$$

from which

$$\|P_\ell^- \varphi\|^2 = \int |\alpha(\lambda)g(\lambda)|^2 f_+(\lambda)\, d\lambda \qquad (2.47)$$

This implies (2.44).

   If

$$W(f, g) = a_0(g_1 f_0 - f_1 g_0) \qquad (2.48)$$

then (2.43) implies

$$\alpha(\lambda) = \frac{W(u_\cdot^+(\lambda + i0), \overline{u_\cdot^-(\lambda + i0)})}{\overline{W(u_\cdot^+(\lambda + i0), u_\cdot^-(\lambda + i0))}} \qquad (2.49)$$

   Since

$$u_0^\pm = 1 \qquad u_1^+ = -a_0 m_0^+ \qquad u_1^- = -(a_0 m_0^-)^{-1} \qquad (2.50)$$

(2.49) implies (1.27). $\square$

3. The Schrödinger Case

In this section, we consider a Schrödinger operator on $\mathbb{R}$,

$$H = -\frac{d^2}{dx^2} + V(x) \qquad (3.1)$$

where V is in L_{loc}^1 and limit point at both $+\infty$ and $-\infty$, so H is the usual selfadjoint operator (see, e.g., [14, Appendix A]). Because it is limit point, there are, for any $z \in \mathbb{C}_+$, unique solutions $u_\pm(x, z)$ obeying

$$-u'' + Vu = zu \qquad (3.2)$$

$$u_\pm(0, z) = 1 \qquad (3.3)$$

$$u_\pm \in L^2(0, \pm\infty) \tag{3.4}$$

For Lebesgue a.e. $\lambda \in \mathbb{R}$,

$$\lim_{\varepsilon \downarrow 0} u_\pm(x, \lambda + i\varepsilon) \equiv u_\pm(x, \lambda + i0) \tag{3.5}$$

exists for all $x \in \mathbb{R}$. Moreover, $\Sigma_{\text{ac}}^{(2)}$, the a.c. spectrum of multiplicity 2, is determined by

$$\text{Im}(\mp u_\pm'(0, \lambda + i0)) > 0 \tag{3.6}$$

(it is always ≥ 0) for a.e. $x \in \Sigma_{\text{ac}}^{(2)}$, that is, positivity for both u_+ and u_-.

The Weyl m-functions (see [14, Appendix A]) are defined by

$$m^\pm(x, \lambda + i0) = \mp \left[\frac{u'(x, \lambda + i0)}{u(x, \lambda + i0)} \right] \tag{3.7}$$

and for $\lambda \in \mathbb{C}_+$ if $\lambda + i0$ is replaced by λ. We define $m(\lambda) \equiv m(x = 0, \lambda)$. The Green's function is given by (for $x \leq y$)

$$G(x, y; \lambda) = \frac{u_-(x, \lambda)u_+(x, \lambda)}{W(\lambda)} \tag{3.8}$$

where

$$W(\lambda) = u_-(x, \lambda)u_+'(x, \lambda) - u_-'(x, \lambda)u_+(x, \lambda) \tag{3.9}$$

is x-independent so that

$$W(\lambda) = -(m^+(\lambda) + m^-(\lambda)) \tag{3.10}$$

and

$$G(x, x; \lambda) = -(m^+(x, \lambda) + m^-(x, \lambda))^{-1} \tag{3.11}$$

H is called *spectrally reflectionless* on $\mathfrak{e} \subset \Sigma_{\text{ac}}^{(2)}$ if and only if for a.e. $\lambda \in \mathfrak{e}$ and all x,

$$m^+(x, \lambda + i0) = -\overline{m^-(x, \lambda + i0)} \tag{3.12}$$

As proven in Davies–Simon [7], if $\chi_y^\pm$ is the characteristic function of $[y, \pm\infty)$, then

$$P_\ell^\pm = \operatorname*{s-lim}_{t \to \mp\infty} e^{itH} \chi_y^- e^{-itH} P_{\text{ac}} \tag{3.13}$$

exists and is y-independent. Indeed, χ_y^- can be replaced by any continuous function, j, which goes to 1 at $-\infty$ and 0 at $+\infty$. If χ_y^- is replaced by χ_y^+, we get $P_r^\pm$. If $\mathcal{H}_{\ell,r}^\pm$ is $\text{Ran}(P_{\ell,r}^\pm)$, then (1.7) and (1.8) hold. If (1.9) and (1.10) hold, we say H is dynamically reflectionless on $\mathcal{H}$.

Following [7], the dynamic reflection probability is given by (1.19)/(1.20) with J replaced by H. The spectral reflection probability (see, e.g., Gesztesy–Nowell–Pötz [13] or Gesztesy–Simon [15]) is given on $\Sigma_{\mathrm{ac}}^{(2)}$ by

$$\left| \frac{m^+(\lambda + i0) + \overline{m^-(\lambda + i0)}}{m^+(\lambda + i0) + m^-(\lambda + i0)} \right|^2 \tag{3.14}$$

Our main theorems in this case are:

Theorem 3.1. *H is dynamically reflectionless on $\mathfrak{e} \in \Sigma_{\mathrm{ac}}^{(2)}$ if and only if it is spectrally reflectionless.*

Theorem 3.2. *$R(\lambda)$ is given by (3.14).*

The proofs closely follow those of Section 2, so we settle for a series of remarks explaining the differences:

1. S is now defined as

$$S(x, y; \lambda) = \pi^{-1} \operatorname{Im} G(x, y; \lambda + i0) \tag{3.15}$$

and there is still a Stone formula like (2.2). One defines

$$f_\pm(\lambda) = \frac{\operatorname{Im} m^\pm(\lambda + i0)}{\pi |m^+(\lambda + i0) + m^-(\lambda + i0)|^2} \tag{3.16}$$

One proves

$$\begin{aligned} S(x, y; \lambda) = {} & \overline{u_+(x, \lambda + i0)}\, u_+(y, \lambda + i0) f_+(\lambda) \\ & + \overline{u_-(x, \lambda + i0)}\, u_-(y, \lambda + i0) f_-(\lambda) \end{aligned} \tag{3.17}$$

The proof is the same as that of Theorem 2.1, except sums over k become integrals over $w \in \mathbb{R}$.

2. Once one has (3.17), one can develop eigenfunction expansions analogously to Theorem 2.2. The one difference is that since $\delta(x)$ is not in L^2, we do not have the analog of (2.23). However,

$$\operatorname{Im} G(x, x; \lambda = i) = \int \frac{\operatorname{Im} G(x, x; \lambda + i0)}{\lambda^2 + 1}\, d\lambda \tag{3.18}$$

which implies that

$$\int_{\Sigma_{\mathrm{ac}}^{(2)}} \frac{|u^\pm(x, \lambda + i0)|^2}{\lambda^2 + 1}\, f_\pm(\lambda)\, d\lambda < \infty \tag{3.19}$$

and that suffices to define an inverse transform on $L^2(\Sigma_{\mathrm{ac}}^{(2)}, d\lambda)$ functions of compact support.

3. As a preliminary to the next step, we note that if η is a function of compact support with a continuous derivative and q is C^∞, then by an integration by parts,

$$\int \eta(x) \left[q(x) \frac{d}{dx} \eta(x) + \frac{d}{dx} (q\eta)(x) \right] dx = 0 \qquad (3.20)$$

4. In computing $(P_\ell^+ \varphi)(x_0)$ for $x_0 < 0$, we can compute $\lim_{t\to\infty} (e^{itH} j e^{-itH} \varphi)$ with a $C^\infty j$ which is 1 if $x < 0$ and 0 if $x > 1$. Thus, in following the calculation in the proof of Theorem 2.3, we start with

$$(P_\ell^- \varphi)(x_0) = \varphi(x_0) - i \lim_{\varepsilon \downarrow 0} \int_{-\infty}^0 e^{\varepsilon s} (e^{isH}[H, j]\varphi)(x_0) \, ds \qquad (3.21)$$

Since $[H, j]$ involves j' and j'', we can instead write $F[H, j]F$ where F is multiplication by a C^∞ function supported in $(x_0, 2)$ which is 1 on $[0, 1]$. When we put in the eigenfunction expansion, we get

$$\int u^-(x_0, \lambda + i0) f_+(\lambda) g(\lambda) h(\lambda) \, d\lambda \qquad (3.22)$$

where h has the form of the left side of (3.20) with

$$\eta(x) = F(x) u^+(x, \lambda + i0) \qquad q(x) = -j'(x) \qquad (3.23)$$

yielding $(P_\ell^- \varphi)(x_0) = \varphi(x_0)$ for $x_0 < 0$. By shifting j to the right, we get this for all x_0 (as in the remark following the proof of Theorem 2.3).

4. The CMV Case

The basic objects in this section are two-sided CMV matrices, $\mathcal{C}$, depending on a sequence $\{\alpha_n\}_{n=-\infty}^\infty$ of Verblunsky coefficients. One-sided CMV matrices appeared first in the numeric matrix literature [1, 2, 33] and were rediscovered by the OPUC community [5]. Two-sided CMV matrices were defined first in [40], although related objects appeared earlier in [3, 10]. For further study, we mention [39, 17, 19, 4].

$\mathcal{C}$ is defined as follows. Given $\alpha \in \mathbb{D}$, we let $\rho = (1 - |\alpha|^2)^{1/2}$ and we let $\Theta(\alpha)$ be the 2×2 matrix,

$$\Theta(\alpha) = \begin{pmatrix} -\alpha & \rho \\ \rho & \bar{\alpha} \end{pmatrix}$$

and let Θ_j be Θ acting on δ_{j-1}, δ_j in $\ell^2(\mathbb{Z})$. Then

$$\mathcal{C} = \mathcal{L}\mathcal{M} \qquad (4.1)$$

where

$$\mathcal{L} = \bigoplus_{n=-\infty}^\infty \Theta_{2n}(\alpha_{2n}) \qquad \mathcal{M} = \bigoplus_{n=-\infty}^\infty \Theta_{2n+1}(\alpha_{2n+1}) \qquad (4.2)$$

First, one can develop a unitary analog of the Davies–Simon theory [7]. It is not hard to show that the Pearson theorem on two-space scattering (see, e.g., [35, Thm. XI.7]) extends to the unitary case. That is, if U and V are unitary, J is bounded, and $UJ - JV$ is trace class, then

$$\text{s-lim}_{t\to\mp\infty} U^{-n}JV^n P_{\text{ac}}(V) \tag{4.3}$$

exists. Thus, if $\chi_n^{\pm}$ are defined as in Section 1, one defines

$$P_\ell^{\pm} = \text{s-lim}_{n\to\mp\infty} \mathcal{C}^{-n}\chi_0^- \mathcal{C}^n P_{\text{ac}}(\mathcal{C})$$
$$P_r^{\pm} = \text{s-lim}_{n\to\pm\infty} \mathcal{C}^{-n}\chi_0^+ \mathcal{C}^n P_{\text{ac}}(\mathcal{C}) \tag{4.4}$$

As in Section 1, we define

$$\mathcal{H}_{\ell,r}^{\pm} = \text{Ran}(P_{\ell,r}^{\pm}) \tag{4.5}$$

and we say $\mathcal{C}$ is *dynamically reflectionless* on $\mathfrak{e}$ if (1.9) and (1.10) hold.

If α_{n-1} is replaced by 1, the CMV matrix breaks into a direct sum of two CMV matrices, $\mathcal{C}_n^+$ on $\ell^2(\{n, n+1, \dots\})$ and $\mathcal{C}_{n-1}^-$ on $\ell^2(\{n-2, n-3, \dots\})$. $F_+(z, n)$ is defined for $z \notin \partial\mathbb{D}$ by setting

$$F_+(z, n) = \left\langle \delta_n, \left(\frac{\mathcal{C}_n^+ + z}{\mathcal{C}_n^+ - z}\right)\delta_n \right\rangle \tag{4.6}$$

and $F_-(z, n-1)$ by

$$F_-(z, n-1) = \left\langle \delta_{n-1}, \left(\frac{\mathcal{C}_{n-1}^- + z}{\mathcal{C}_{n-1}^- - z}\right)\delta_{n-1} \right\rangle \tag{4.7}$$

It is known (see, e.g., [40]) that when restricted to $z \in \mathbb{D}$, $F_+(z, n)$ is the Carathéodory function whose Verblunsky coefficients are $\{\alpha_n, \alpha_{n+1}, \dots\}$ and $F_-(z, n-1)$ has Verblunsky coefficients $\{-\bar{\alpha}_{n-2}, -\bar{\alpha}_{n-3}, \dots\}$. We will let $F_\pm(z) = F_\pm(z, n = 0)$.

As Carathéodory functions, $F_\pm(z, n)$ have a.e. boundary values on $\partial\mathbb{D}$ which we denote by $F_\pm(e^{i\theta}, n) = \lim_{r\uparrow 1} F_\pm(re^{i\theta}, n)$. $\mathcal{C}$ is called *spectrally reflectionless* on $\mathfrak{e} \subset \partial\mathbb{D}$ if and only if for a.e. $e^{i\theta} \in \mathfrak{e}$ and all $n \in \mathbb{Z}$,

$$F_+(e^{i\theta}, n) = \overline{F_-(e^{i\theta}, n)} \tag{4.8}$$

There is an equivalent definition using Schur functions (see, e.g., [4]). The equivalence is an easy computation using the relations between the Carathéodory and Schur functions (see, e.g., [17]). It is known [17] that (4.8) for one n implies it for all n. It is also known [4] that while (4.8) implies $\langle \delta_n, (\mathcal{C} + z)/(\mathcal{C} - z)\delta_n \rangle$ has purely real boundary values a.e. on $\mathfrak{e}$, the converse can be false.

The dynamic reflection probability $R(e^{i\theta})$ is given by $(1.19)/(1.20)$ with J replaced by $\mathcal{C}$. The spectral reflection probability is given on $\Sigma_{\mathrm{ac}}^{(2)}$ by

$$\left| \frac{F_+(e^{i\theta}) - \overline{F_-(e^{i\theta})}}{F_+(e^{i\theta}) + F_-(e^{i\theta})} \right|^2 \tag{4.9}$$

Our main theorems in this case are:

Theorem 4.1. *$\mathcal{C}$ is dynamically reflectionless on $\mathfrak{e}$ if and only if it is spectrally reflectionless on $\mathfrak{e}$.*

Theorem 4.2. *$R(e^{i\theta})$ is given by (4.9).*

The proofs closely follow those of Section 2, so we again settle for a series of remarks:

1. The analysis requires us to simultaneously study solutions of $\mathcal{C}$ and $\mathcal{C}^T$. To do so, let

$$\mathcal{E} = \begin{pmatrix} \mathcal{C} & 0 \\ 0 & \mathcal{C}^T \end{pmatrix}$$

acting on two sequences labeled by all of $\mathbb{Z}$. Following Gesztesy–Zinchenko [19], let

$$\begin{pmatrix} p(z,n) \\ r(z,n) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q(z,n) \\ s(z,n) \end{pmatrix}$$

be the two (Laurent polynomial) solutions to the equation

$$\mathcal{E} \begin{pmatrix} u \\ v \end{pmatrix} = z \begin{pmatrix} u \\ v \end{pmatrix} \tag{4.10}$$

satisfying the initial conditions

$$\begin{pmatrix} p(z,0) \\ r(z,0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q(z,0) \\ s(z,0) \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

That is, for one solution, the components of u are p and of v are r, and this solution is uniquely determined by the initial conditions given (see (4.12)). Similarly, the components for the second solution are given by q and s. Finally, we let

$$\begin{pmatrix} u_\pm(z,n) \\ v_\pm(z,n) \end{pmatrix} = \begin{pmatrix} q(z,n) \\ s(z,n) \end{pmatrix} \pm F_\pm(z) \begin{pmatrix} p(z,n) \\ r(z,n) \end{pmatrix}$$

be the unique solutions that are ℓ^2 at $\pm\infty$, normalized by

$$\begin{pmatrix} u_\pm(z,0) \\ v_\pm(z,0) \end{pmatrix} = \begin{pmatrix} -1 \pm F_\pm(z) \\ 1 \pm F_\pm(z) \end{pmatrix} \tag{4.11}$$

We note that there are a number of relations between $u_\pm$ and $v_\pm$ that we will need (see [19]). First, (4.10) is equivalent to

$$\begin{pmatrix} u(z,n) \\ v(z,n) \end{pmatrix} = T(z,n) \begin{pmatrix} u(z,n-1) \\ v(z,n-1) \end{pmatrix} \tag{4.12}$$

where

$$T(z,n) = \begin{cases} \dfrac{1}{\rho_n} \begin{pmatrix} \alpha_n & z \\ 1/z & \overline{\alpha_n} \end{pmatrix}, & n \text{ odd} \\[2em] \dfrac{1}{\rho_n} \begin{pmatrix} \overline{\alpha_n} & 1 \\ 1 & \alpha_n \end{pmatrix}, & n \text{ even} \end{cases} \tag{4.13}$$

Similarly, (4.10) implies

$$\begin{pmatrix} u(z,2n-1) \\ u(z,2n) \end{pmatrix} = \Theta_{2n}(\alpha_{2n}) \begin{pmatrix} v(z,2n-1) \\ v(z,2n) \end{pmatrix}$$
$$\begin{pmatrix} u(z,2n-2) \\ u(z,2n-1) \end{pmatrix} = \Theta_{2n-1}(\alpha_{2n-1}) \begin{pmatrix} v(z,2n-2) \\ v(z,2n-1) \end{pmatrix} \tag{4.14}$$

Finally, for all $n \in \mathbb{Z}$, we have

$$v_\pm(1/\bar{z},n) = -\overline{u_\pm(z,n)} \tag{4.15}$$

This is because

$$\mathcal{C}u = zu \text{ holds} \quad \Leftrightarrow \quad \mathcal{C}^T \bar{u} = (1/\bar{z})\bar{u} \text{ holds}$$

and (4.6)/(4.7) imply $F_\pm(1/\bar{z}) = -\overline{F_\pm(z)}$, and because the solutions to (4.10) that are ℓ^2 at $\pm\infty$ are unique up to normalization.

2. Using the solutions $u_\pm(z,n)$ and $v_\pm(z,n)$ we can write the analog of (1.28) (see [19]):

$$(\mathcal{C} - z)^{-1}_{nm} = \frac{-1}{zW(z)} \begin{cases} u_-(z,n)v_+(z,m), & n < m \text{ or } n = m = 2k+1 \\ v_-(z,m)u_+(z,n), & m < n \text{ or } n = m = 2k \end{cases} \tag{4.16}$$

where

$$W(z) = u_+(z,n)v_-(z,n) - v_+(z,n)u_-(z,n) \tag{4.17}$$

is independent of $n \in \mathbb{Z}$.

3. Next we find the analog of $[J, \chi_0^+]$. Due to the structure of (4.1), the results are different depending on whether n is even or odd. For n even:

$$[\mathcal{C}, \chi_n^+] = -\rho_n \big(\rho_{n-1}|\delta_n\rangle\langle\delta_{n-2}| + \overline{\alpha_{n-1}}|\delta_n\rangle\langle\delta_{n-1}| + \alpha_{n+1}|\delta_{n-1}\rangle\langle\delta_n| - \rho_{n+1}|\delta_{n-1}\rangle\langle\delta_{n+1}| \big) \tag{4.18}$$

while if n is odd we get the same thing but transposed and with a minus sign:

$$[\mathcal{C}, \chi_n^+] = \rho_n\big(\rho_{n-1}|\delta_{n-2}\rangle\langle\delta_n| + \overline{\alpha_{n-1}}|\delta_{n-1}\rangle\langle\delta_n| + \alpha_{n+1}|\delta_n\rangle\langle\delta_{n-1}| - \rho_{n+1}|\delta_{n+1}\rangle\langle\delta_{n-1}|\big)$$
(4.19)

4. S is defined (using a.e. boundary values) as

$$S(n, m; e^{i\theta}) = \frac{1}{2\pi}\lim_{r\uparrow 1}((\mathcal{C}+re^{i\theta})(\mathcal{C}-re^{i\theta})^{-1} - (\mathcal{C}+r^{-1}e^{i\theta})(\mathcal{C}-r^{-1}e^{i\theta})^{-1})_{nm}$$
(4.20)

and there is a Stone formula like (2.2). Proceeding as in Section 2 and using (4.15) and (4.16), one can deduce the analog of (2.8):

$$S(n, m; e^{i\theta}) = u_+(e^{i\theta}, n)\overline{u_+(e^{i\theta}, m)}f_+(e^{i\theta}) + u_-(e^{i\theta}, n)\overline{u_-(e^{i\theta}, m)}f_-(e^{i\theta})$$
(4.21)

where $u_\pm(e^{i\theta}, n) = \lim_{r\uparrow 1} u_\pm(re^{i\theta}, n)$ and

$$\pi f_\pm(e^{i\theta}) = \lim_{r\uparrow 1}\frac{1}{re^{i\theta}|W(re^{i\theta})|^2}\langle u_\mp(r^{-1}e^{i\theta}), [\mathcal{C}, \chi_k^\mp]u_\mp(r^{-1}e^{i\theta})\rangle$$

As before, this is independent of k, and choosing $k = 0$ one may use (4.11), (4.12), and (4.18)/(4.19) to find the analog of (2.3):

$$f_\pm(e^{i\theta}) = \frac{4\,\mathrm{Re}\,F_\mp(e^{i\theta})}{\pi|W(e^{i\theta})|^2}$$
(4.22)

Once one has (4.21), one may develop eigenfunction expansions exactly as in Theorem 2.2.

5. To prove Theorem 4.1, we first define $P^\pm$ and $\overline{P^\pm}$ as in Section 2 but using $\lim_{r\uparrow 1} u_\pm(re^{i\theta}, n)$ and $\lim_{r\uparrow 1} u_\pm(r^{-1}e^{i\theta}, n)$ respectively. As before, we consider $P_\ell^+ = \text{s-}\lim_{n\to-\infty}\mathcal{C}^{-n}\chi_0^-\mathcal{C}^n P_{\mathrm{ac}}(\mathcal{C})$. Because

$$\mathcal{C}^{-n}\chi_0^-\mathcal{C}^n - \mathcal{C}^{-(n-1)}\chi_0^-\mathcal{C}^{n-1} = \mathcal{C}^{-n}[\chi_0^-, \mathcal{C}]C^{n-1}$$

and the strong limit defining P_ℓ^+ exists, we see

$$P_\ell^+ = \chi_0^- + \text{s-}\lim_{n\to-\infty}\sum_{k=1}^{n}\mathcal{C}^{-k}[\chi_0^-, \mathcal{C}]\mathcal{C}^{k-1}$$

Choosing a dense set of $\varphi \in \mathrm{Ran}(P^+)$ as before, and using the eigenfunction expansion and an abelian theorem, we find

$$(P_\ell^+\varphi)_m = (\chi_0^-\varphi)_m + \left(\lim_{n\to-\infty}\sum_{k=1}^{n}\mathcal{C}^{-k}[\chi_0^-, \mathcal{C}]\mathcal{C}^{k-1}\varphi\right)_m$$

$$= (\chi_0^-\varphi)_m + \lim_{r\uparrow 1}\lim_{n\to-\infty}\int\sum_{k=1}^{n}\mathcal{C}^{-k}[\chi_0^-, \mathcal{C}](re^{i\theta})^{k-1}u_+(e^{i\theta}, m)g(e^{i\theta})f_+(e^{i\theta})\frac{d\theta}{2\pi}$$

$$= (\chi_0^- \varphi)_m + \int (\mathcal{C} - e^{i\theta})^{-1} [\chi_0^-, \mathcal{C}] u_+(e^{i\theta}, m) g(e^{i\theta}) f_+(e^{i\theta}) \frac{d\theta}{2\pi}$$

The proof of Theorem 4.1 then proceeds exactly as the proof of Theorem 1.3, but now using (4.11)–(4.22). The proof of Theorem 4.2 follows that of Theorem 2.4 but with $\lim_{r\uparrow 1} u_\pm(re^{i\theta}, n)$ and $\lim_{r\uparrow 1} u_\pm(r^{-1}e^{i\theta}, n)$ replacing $u_\pm(x + i0, n)$ and $\overline{u_\pm(x + i0, n)}$.

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